

On (n, d) -perfect rings

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Abstract

In this paper, we introduce the notion of “ (n, d) -perfect rings” which is in some way a generalization of the notion of “ S -rings”. After we give some basic results of this rings and we survey the relationship between “ $A(n)$ property” and “ (n, d) -perfect property”. Finally, we investigate the “ (n, d) -perfect property” in pullback rings.

Key Words. (n, d) -perfect ring, $A(n)$ ring, n -presented, homological dimensions, pullback ring.

1 Introduction

The object of this paper is to introduce a doubly filtered set of classes of rings which may serve to shed further light on the structures of non-Noetherian rings. Throughout this work, all rings are commutative with identity element, and all modules are unitary. By a “local” ring we mean a (not necessarily Noetherian) ring with a unique maximal ideal.

The classes of rings we will define here are in some ways generalizations of the notion of “ $A(n)$ rings” which is introduced by Cox and Pendleton in [4].

Let R be a ring and let M be an R -module. As usual we use $\text{pd}_R(M)$ and $\text{fd}_R(M)$ to denote the usual projective and flat dimensions of M , respectively. The classical global and weak dimension of R are respectively $\text{gldim}(R)$ and $\text{wdim}(R)$. If R is an integral domain, we denote

its quotient field by $\text{qf}(R)$.

We say that M is n -presented R -module whenever there is an exact sequence $F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ of R -modules in which each F_i is a free finitely generated R -module. In particular, 0-presented and 1-presented R -modules are respectively finitely generated and finitely presented R -modules. We recall that a coherent ring is a ring such that each finitely generated ideal is finitely presented.

As in [2, 15], we set $\lambda_R(M) = \sup\{n/M \text{ is } n\text{-presented}\}$ except that we set $\lambda_R(M) = -1$ if M is not finitely generated. Note that $\lambda_R(M) \geq n$ is a way to express the fact that M is n -presented.

The headings of the various sections indicate their content. Thus in the second section, we present the definition of this classes of rings and (mostly we known) the basic results. The third section is devoted to establish the relationship between the “ $A(n)$ property” and the property of the classes of rings which we will define below, and the fourth section of this survey deal with these classes of rings in pullbacks. Finally, we give an extensive set of references.

General background materials can be found in Rotman [14] (1979), and Glaz [10] (1989).

2 Definition and basic results

In this section we introduce and study the (n, d) -perfect ring which is defined as follows.

Definition 2.1 *Let n and d be a nonnegatives integers. A ring R is said to be an (n, d) -perfect ring, if every n -presented module with flat dimension at most d , has projective dimension at most d .*

We illustrate this notion with the following example. First it is well known that if a flat R -module M is finitely presented, or finitely generated with R either a semilocal ring or an integral domain, then M is projective (see [7, Theorem 2]). And a ring is called an S -ring if every finitely generated flat R -module is projective (see [13]).

Example 2.2 1. R is an S -ring if and only if R is an $(0, 0)$ -perfect ring.
2. If R is a semilocal ring, then R is an (n, n) -perfect ring for every $n \geq 0$.

3. If R is a domain, then R is an (n, n) -perfect ring for every $n \geq 0$.
4. If R is an (n, d) -perfect ring, then R is an (n', d) -perfect ring for every $n' \geq n$.
5. For every $n > d$, R is an (n, d) -perfect ring.
6. If R is a perfect ring, then R is (n, d) -perfect for every $n \geq 0$ and $d \geq 0$.

Proof. Obvious.

The following proposition give two results concerning the Noetherian rings and coherent rings.

- Proposition 2.3** 1. If R is a Noetherian ring, then R is an (n, d) -perfect ring for every $n \geq 0$ and $d \geq 0$.
2. If R is a coherent ring, then R is an (n, d) -perfect ring for every $n \geq 1$ and $d \geq 0$.

Proof. Obvious.

Furthermore, we construct an example of ring which it is a $(0, 1)$ -perfect ring and which is not a $(0, 0)$ -perfect ring (Example 2.4). Also we exhibit an example of a $(1, 1)$ -perfect ring which is not a $(0, 1)$ -perfect ring (Example 2.5).

Example 2.4 Let R be an hereditary and Von Neumann regular ring which it is not semi-simple. Then R is a $(0, 1)$ -perfect ring which is not a $(0, 0)$ -perfect ring.

Proof. The ring R is a $(0, 1)$ -perfect ring since R is an hereditary ring. If R is a $(0, 0)$ -perfect ring hence every finitely generated R -module is projective since R is a Von Neumann regular ring, hence we have a contradiction with R is not semi-simple.

Example 2.5 Let R be a non-Noetherian Prüfer domain. Then R is a $(1, 1)$ -perfect domain which it is not a $(0, 1)$ -perfect domain.

Proof. The ring R is a $(1, 1)$ -perfect ring since R is a domain. On the other hand, we show that R is not a $(0, 1)$ -perfect ring. Let I be a not finitely generated ideal of R (since R is not Noetherian), then I is not projective. Of course I is flat since $\text{wdim}(R) \leq 1$. Thus R/I is 0-presented with $\text{fd}_R(R/I) \leq 1$ and $\text{pd}_R(R/I) \geq 2$ as desired.

Next we give a homological characterization of (n, d) -perfect ring.

Theorem 2.6 *Let R be a commutative ring. Then the following are equivalent.*

1. R is an (n, d) -perfect ring.
2. $\text{Ext}_R^{d+1}(M, N) = 0$ for all R -modules M, N such that $\lambda_R(M) \geq n$, $\text{fd}_R(M) \leq d$ and $\text{fd}_R(N) \leq d$.
3. $\text{Ext}_R^{d+1}(M, N) = 0$ for all R -modules M, N such that $\lambda_R(M) \geq n$, $\lambda_R(N) \geq n - (d + 1)$, $\text{fd}_R(M) \leq d$ and $\text{fd}_R(N) \leq d$.

The proof of this Theorem involves the following Lemmas.

Lemma 2.7 *Let R be a ring, and let M be an n -presented flat R -module, where $n \geq 0$. Then M is projective if and only if $\text{Ext}_R^1(M, N) = 0$ for all R -modules N such that $\lambda_R(N) \geq n - 1$ and N is a flat R -module.*

Proof. Necessity is clear. To proof sufficiency, let $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ be an exact sequence with F a finitely generated free R -module. Then K is an $(n - 1)$ -presented flat R -module, hence by hypothesis $\text{Ext}_R^1(M, K) = 0$. It follows that the exact sequence splits, making M a direct summand of F . Therefore M is a projective R -module.

Lemma 2.8 *Let R be a ring, and let M be an n -presented R -module such that $\text{fd}_R(M) \leq d$. Then $\text{pd}_R(M) \leq d$ if and only if $\text{Ext}_R^{d+1}(M, N) = 0$ for all R -modules N such that $\lambda_R(N) \geq n - (d + 1)$ and $\text{fd}_R(N) \leq d$.*

Proof. This follows from Lemma 2.7 by dimension shifting.

Proof. [Proof of Theorem 2.6] Follows from Lemma 2.8.

Now, in the following we prove that the (n, d) -perfect property descends into a faithfully flat ring homomorphism.

Theorem 2.9 *Let $R \longrightarrow S$ be a ring homomorphism making S a faithfully flat R -module. If S is an (n, d) -perfect ring then R is an (n, d) -perfect ring.*

Proof. Let M be an n -presented R -module with $\text{fd}_R(M) \leq d$. Our aim is to show that $\text{pd}_R(M) \leq d$. We have $\lambda_S(M \otimes_R S) \geq n$ and $\text{fd}_S(M \otimes_R S) \leq d$ since S is a flat R -module, so $\text{pd}_S(M \otimes_R S) \leq d$ since S is an (n, d) -perfect ring.

Let $0 \longrightarrow P \longrightarrow F_{d-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ be an exact sequence of R -modules, where F_i is a free R -module for each i and P is a flat A -module. Thus $P \otimes_R S$ is a projective S -module. By [11, Example 3.1.4. page 82] P is a projective R -module.

We use this result to study the “ (n, d) -perfect property” of some particular rings.

Corollary 2.10 1. Let $A \subset B$ be two rings such that B is a flat A -module. Let $S = A + XB[X]$, where X is an indeterminate over B . If S is an (n, d) -perfect ring, then so is A .

2. Let R be a ring and X is an indeterminate over R . If $R[X]$ is an (n, d) -perfect ring then so is R .

Proof. 1) The ring B is a flat A -module and $XB[X] \cong B[X]$ thus $S = A + XB[X]$ is a faithfully flat A -module. By Theorem 2.9 the ring A is an (n, d) -perfect since S is an (n, d) -perfect ring.

2) Similarly, by Theorem 2.9, since $R[X]$ is a faithfully flat R -module.

We close this section by establishing the transfer of the (n, d) -perfect property to finite direct product.

Theorem 2.11 Let $(R_i)_{i=1, \dots, m}$ be a family of rings. Then $\prod_{i=1}^m R_i$ is an (n, d) -perfect ring if and only if R_i is an (n, d) -perfect ring for each $i = 1, \dots, m$.

The proof of this Theorem involves the following results.

Lemma 2.12 [12, Lemma 2.5.] Let $(R_i)_{i=1,2}$ be a family of rings and E_i be an R_i -module for $i = 1, 2$. We have :

$$1. \text{pd}_{R_1 \times R_2}(E_1 \times E_2) = \sup\{\text{pd}_{R_1}(E_1), \text{pd}_{R_2}(E_2)\}.$$

$$2. \lambda_{R_1 \times R_2}(E_1 \times E_2) = \inf\{\lambda_{R_1}(E_1), \lambda_{R_2}(E_2)\}$$

Lemma 2.13 Let $(R_i)_{i=1,2}$ be a family of rings and E_i be an R_i -module for $i = 1, 2$. We have : $\text{fd}_{R_1 \times R_2}(E_1 \times E_2) = \sup\{\text{fd}_{R_1}(E_1), \text{fd}_{R_2}(E_2)\}$.

Proof. This proof is analogous to the proof of Lemma 2.12 (1). **Proof.** [Proof of Theorem 2.11] We use induction on m , it suffices to prove the assertion for $m = 2$. Let R_1 and R_2 be two rings such that $R_1 \times R_2$ is an (n, d) -perfect ring.

Let E_1 be an R_1 -module such that $\text{fd}_{R_1}(E_1) \leq d$, $\lambda_{R_1}(E_1) \geq n$ and let E_2 be an R_2 -module such that $\text{fd}_{R_2}(E_2) \leq d$, $\lambda_{R_2}(E_2) \geq n$. By Lemma 2.13. $\text{fd}_{R_1 \times R_2}(E_1 \times E_2) = \sup\{\text{fd}_{R_1}(E_1), \text{fd}_{R_2}(E_2)\}$. And by Lemma 2.12(2) $\lambda_{R_1 \times R_2}(E_1 \times E_2) = \inf\{\lambda_{R_1}(E_1), \lambda_{R_2}(E_2)\}$. Thus $\lambda_{R_1 \times R_2}(E_1 \times E_2) \geq n$ and $\text{fd}_{R_1 \times R_2}(E_1 \times E_2) \leq d$. So $\text{pd}_{R_1 \times R_2}(E_1 \times E_2) \leq d$ since $R_1 \times R_2$ is an (n, d) -perfect ring. By Lemma 2.12(1) $\text{pd}_{R_1 \times R_2}(E_1 \times E_2) = \sup\{\text{pd}_{R_1}(E_1), \text{pd}_{R_2}(E_2)\}$. Thus $\text{pd}_{R_1}(E_1) \leq n$ and $\text{pd}_{R_2}(E_2) \leq n$. Therefore R_1 is an (n, d) -perfect ring and R_2 is an (n, d) -perfect ring.

Conversely, let R_1 and R_2 be two (n, d) -perfect rings and let $E_1 \times E_2$ be an $R_1 \times R_2$ -module where E_i is an R_i -module for each $i = 1, 2$, such that $\text{fd}_{R_1 \times R_2}(E_1 \times E_2) \leq d$ and $\lambda_{R_1 \times R_2}(E_1 \times E_2) \geq n$. By Lemma 2.12, $\lambda_{R_1}(E_1) \geq n$, $\lambda_{R_2}(E_2) \geq n$ and by Lemma 2.13, $\text{fd}_{R_1}(E_1) \leq d$, $\text{fd}_{R_2}(E_2) \leq d$, then $\text{pd}_{R_1}(E_1) \leq d$ and $\text{pd}_{R_2}(E_2) \leq d$, since R_1 and R_2 are an (n, d) -perfect rings. By Lemma 2.12, $\text{pd}_{R_1 \times R_2}(E_1 \times E_2) \leq d$. Therefore $R_1 \times R_2$ is an (n, d) -perfect rings

3 Relationship between the $A(n)$ property and (n, d) -perfect property

The purpose of the present section is to establish a natural bridge between $A(n)$ rings and (n, d) -perfect rings.

First, we recall the definition of the $A(n)$ rings introduced in [4].

Definition 3.1 [4, page 139] *Let n be nonnegative integer. A ring R is said to be an $A(n)$ ring if given any exact sequence $0 \longrightarrow M \longrightarrow E_1 \longrightarrow \dots \longrightarrow E_n$ of finitely generated R -modules with M flat and E_i free for each i , then M is projective.*

We show in the next Theorem the existence of relationship between the “ $A(n)$ property” and “ (n, d) -perfect property”.

Theorem 3.2 *A ring R is an $A(n)$ ring if and only if R is an (n, n) -perfect ring.*

Proof. Assume that R is an $A(n)$ ring and let M be an R -module such that $\lambda_R(M) \geq n$ and $\text{fd}_R(M) \leq n$. Then there exist an exact sequence $0 \longrightarrow P \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ of finitely generated

R -modules with P flat and F_i free for each i . Then P is projective since R is an $A(n)$ ring. Therefore R is an (n, n) -perfect ring.

Conversely, assume that R is an (n, n) -perfect ring. Let $0 \longrightarrow M \longrightarrow F_1 \longrightarrow \dots \xrightarrow{u_n} F_n$ be an exact sequence of finitely generated R -modules with M flat and F_i free for each i . We show that M is projective. The exact sequence

$$0 \longrightarrow M \longrightarrow F_1 \longrightarrow \dots \xrightarrow{u_n} F_n \longrightarrow \text{Coker } u_n \longrightarrow 0$$

show that $\lambda_R(\text{Coker } u_n) \geq n$ and $\text{fd}_R(\text{Coker } u_n) \leq n$. Hence, $\text{pd}_R(\text{Coker } u_n) \leq n$ since R is an (n, n) -perfect ring and so M is projective. Therefore R is an $A(n)$ ring.

The Theorem 3.2 combined with the results obtained by Cox and Pendleton [4] may be used to find several corollaries.

Corollary 3.3 *Let $\varphi : R \hookrightarrow T$ be an injective ring homomorphism.*

1. *If T is a $(0, 0)$ -perfect ring so is R .*
2. *If T is a (n, n) -perfect ring ($n \geq 1$) and T is a flat R -module then R is an (n, n) -perfect ring.*

Proof. By [4, Theorem 2.4] and Theorem 3.2.

Theorem 3.4 *A ring R is a $(1, 1)$ -perfect ring if and only if each pure ideal of R which is the annihilator of a finitely generated ideal of R is generated by an idempotent.*

Proof. By [4, Theorem 3.8] and Theorem 3.2.

The next example gives a $(1, 1)$ -perfect ring R and a multiplicative set S of R such that $S^{-1}R$ is not a $(1, 1)$ -perfect ring.

Example 3.5 [4, Example 5.17] *Let $R = \mathbb{Z}[f, x_1, x_2, \dots]$, with defining relations $fx_i(1 - x_j) = 0$, $1 \leq i < j$ and $2fx_i = 0$, $1 \leq i$. Put $S = \{f^n/n \geq 1\}$. Then R is an $(1, 1)$ -perfect ring, but $S^{-1}R$ is not a $(1, 1)$ -perfect ring.*

4 Transfer of the (n, d) -perfect property in pullbacks

Pullbacks occupy an important niche in homological algebra because they produce interesting examples (see for example [10, Section 1, Chapter 5]). For a history of pullbacks, see [8, Appendix 2] and [9, pages

582-584].

The following Theorem is the main result of this section.

Theorem 4.1 *Let $A \hookrightarrow B$ be an injective flat ring homomorphism and let Q be a pure ideal of A such that $QB = Q$ and $\lambda_A(Q) \geq n - 1$.*

1. *Assume that B is an (n, d) -perfect ring. Then A/Q is an (n, d) -perfect ring if and only if A is an (n, d) -perfect ring.*
2. *Assume that $B = S^{-1}A$, where S is a multiplicative set of A . Then A is an (n, d) -perfect ring if and only if B and A/Q are an (n, d) -perfect rings.*

Before proving this Theorem, we establish the following Lemmas.

At the start, we recall the notion of flat epimorphism of rings, which is defined as follows: Let $\Phi : A \rightarrow B$ be a ring homomorphism. B (or Φ) is called a flat epimorphism of A , if B is a flat A -module and Φ is an epimorphism, that is, for any two ring homomorphism $B \xrightarrow{f} C$, $C \xrightarrow{g} A$ a ring, satisfying $f \circ \Phi = g \circ \Phi$, we have $f = g$ [10, pages 13-14]. For example $S^{-1}A$ is a flat epimorphism of A for every multiplicative set S of A . Also, the quotient ring A/I is a flat epimorphism of A for every pure ideal I of A , that is, A/I is a flat A -module [10, Theorem 1. 2. 15].

Lemma 4.2 *Let A and B be two rings such that $\Phi : A \rightarrow B$ be a flat epimorphism of A and $\lambda_A(B) \geq n$.*

If A is an (n, d) -perfect ring then B is an (n, d) -perfect ring.

In particular, if A is an (n, d) -perfect ring, then so is the quotient ring A/I for every pure ideal I of A such that $\lambda_A(I) \geq n - 1$ ($n \geq 0$).

Proof. Let M be a B -module such that $\lambda_B(M) \geq n$ and $\text{fd}_B(M) \leq d$. Our aim is to show that $\text{pd}_B(M) \leq d$.

By hypothesis we have $\text{Tor}_k^A(M, B) = 0$ for all $k > 0$. By [3, Proposition 4.1.3], we have for any B -module N

$$(*) \quad \text{Ext}_A^{d+1}(M, N \otimes_A B) \cong \text{Ext}_B^{d+1}(M \otimes_A B, N \otimes_A B)$$

From [10, Theorem 1.2.19] we get $M \otimes_A B \cong M$ and $N \otimes_A B \cong N$.

On the other hand, $\text{fd}_A(M) \leq \text{fd}_B(M) \leq d$ [3, Exercise 10.p.123] and

$\lambda_A(M) \geq n$ [5, Lemma 2.6]. Thus $\text{pd}_A(M) \leq d$ since A is an (n, d) -perfect ring. Hence, $(*)$ implies $\text{Ext}_B^{d+1}(M, N) = 0$, therefore $\text{pd}_B(M) \leq d$.

The Example 3.5 shows that Lemma 4.2 is not true in general without assuming that $\lambda_A(B) \geq n$.

Lemma 4.3 *Let $A \hookrightarrow B$ be an injective flat ring homomorphism and let Q be a pure ideal of A such that $QB = Q$. Let E be an A -module. Then:*

1. $\lambda_A(E) \geq n \Leftrightarrow \lambda_B(E \otimes_A B) \geq n$ and $\lambda_{A/Q}(E \otimes_A A/Q) \geq n$.
2. $\text{fd}_A(E) \leq d \Leftrightarrow \text{fd}_B(E \otimes_A B) \leq d$ and $\text{fd}_{A/Q}(E \otimes_A A/Q) \leq d$.
3. $\text{pd}_A(E) \leq d \Leftrightarrow \text{pd}_B(E \otimes_A B) \leq d$ and $\text{pd}_{A/Q}(E \otimes_A A/Q) \leq d$.

Proof. Similar to that of [6, Lemma 2.4] will be omitted.

Proof. [Proof of Theorem 4.1] 1) If A is an (n, d) -perfect ring since Q is an $(n - 1)$ -presented pure ideal of A , by Lemma 4.2. A/Q is an (n, d) -perfect ring.

Conversely, assume that B and A/Q are an (n, d) -perfect rings. Let M be an A -module such that $\lambda_A(M) \geq n$ and $\text{fd}_A(M) \leq d$.

Then $\lambda_B(M \otimes_A B) \geq n$ and $\text{fd}_B(M \otimes_A B) \leq d$. So $\text{pd}_B(M \otimes_A B) \leq d$ since B is an (n, d) -perfect ring. Also $\lambda_{A/Q}(M \otimes_A A/Q) \geq n$ and $\text{fd}_{A/Q}(M \otimes_A A/Q) \leq d$. So $\text{pd}_{A/Q}(M \otimes_A A/Q) \leq d$ since A/Q is an (n, d) -perfect rings. By Lemma 4.3. $\text{pd}_A(M) \leq n$. Therefore A is an (n, d) -perfect ring.

2) Follows from Lemma 4.2. Lemma 4.3 and 1).

Corollary 4.4 *Let D be an integral domain, $K = \text{qf}(D)$ and let $n \geq 2$, $n \geq 0$ and $d \geq 0$ be a positive integers. Consider the quotient ring $S = K[X]/(X^n - X) = K + \overline{X}K[\overline{X}] = K + I$ with $I = \overline{X}K[\overline{X}]$. Set $R = D + I$. Then R is an (m, d) -perfect ring if and only if D is an (m, d) -perfect ring.*

Proof. First we show that I is a pure ideal of R . Let $u := \overline{X}^i(a_0 + a_1\overline{X} + \dots + a_{n-1}\overline{X}^{n-1})$ be an element of I , where $a_i \in K$ for $1 \leq i \leq n-1$, and $a_0 \neq 0$. Hence $u(1 - \overline{X}^{n-1}) = 0$ $(*)$ since $\overline{X}^i(1 - \overline{X}^{n-1}) = \overline{X}^i - \overline{X}^{n+(i-1)} = \overline{X}^i - \overline{X}^n\overline{X}^{i-1} = \overline{X}^i - \overline{X}^i = \overline{0}$. Therefore, I is a pure ideal of R by [10, Theorem 1.2.15] since $\overline{X}^{n-1} \in I$.

Our aim is to show that $\lambda_R(I) = \infty$. We have $R\overline{X}^{n-1} = (D +$

$\overline{X}K[\overline{X}]\overline{X}^{n-1} = D\overline{X}^{n-1} + \overline{X}K[\overline{X}] = D\overline{X}^{n-1} + I = I$ since $\overline{X}^n = \overline{X}$. We claim that $\text{Ann}_R(I) = R(1 - \overline{X}^{n-1})$. Indeed, by (*) $R(1 - \overline{X}^{n-1}) \subseteq \text{Ann}_R(I)$. Conversely, let $v := d + a_1\overline{X} + \dots + a_{n-1}\overline{X}^{n-1} \in \text{Ann}_R(I)$, where $d \in D$ and $a_i \in K$. Hence $0 = (d + a_1\overline{X} + \dots + a_{n-1}\overline{X}^{n-1})\overline{X}^{n-1} = a_1\overline{X} + \dots + (d + a_{n-1})\overline{X}^{n-1}$ and so $a_1 = a_2 = \dots = a_{n-2} = 0$ and $d + a_{n-1} = 0$. This means, $v = d(1 - \overline{X}^{n-1}) \in R(1 - \overline{X}^{n-1})$ as desired. Also, the same proof as above shows that $\text{Ann}_R(R(1 - \overline{X}^{n-1})) = I$. Therefore, $\lambda_R(I) = \infty$ as desired.

On the other hand, we have S an artinian ring. According to [1, Corollary 28.8] we deduce S is a perfect ring, so S is an (m, d) -perfect ring. We obtain the result by Theorem 4.1 and this completes the proof of Corollary 4.4.

From this Corollary we deduce easily the following example.

Example 4.5 Let D be an integral domain such that $\text{gldim}(D) = d$, let $K = \text{qf}(D)$ and let $n \geq 2$. Consider the quotient ring $S = K[X]/(X^n - X) = K + \overline{X}K[\overline{X}] = K + I$ with $I = \overline{X}K[\overline{X}]$. Set $R = D + I$. Then R is a $(1, d)$ -perfect ring.

Proof. Clear by Corollary 4.4 since D is an $(1, d)$ -perfect ring (since $\text{gldim}(D) = d$).

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